MATH 20C - Allen - Midterm 2

Show all work. No credit might be given for unsupported answers, even if correct.

Note the magenta color is only a suggestion for the partial credit breakdown. Alternative or complex solutions must necessarily take a more personalized approach

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Problem 0 (1 point)

Write your name, PID, and section number on the front of your bluebook.

Solution:

Professor, 53014497, C00 (1 point)

Problem 1 (10 points)

a) The unit vector $\hat{A} = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}, 0\right)$, is tangent to the unit sphere given by $x^2 + y^2 + z^2 = 1$, at the point $(x_0, y_0, z_0) = \left(\frac{1}{4}, \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)$. Find another *unit* tangent vector, $\hat{B}$, orthogonal to $\hat{A}$.

Solution:

$\hat{A}$ is tangent and therefore orthogonal to the normal

$\vec{n} = \nabla\left(x^2 + y^2 + z^2\right) \mid_{(x_0,y_0,z_0)} = (2x, 2y, 2z) \mid_{(x_0,y_0,z_0)} = 2\left(\frac{1}{4}, \frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)$.

(3 points)

A vector orthogonal to both $\hat{A}$ and $\vec{n}$ can be found with the cross product: $\vec{B} = \hat{A} \times \vec{n} = \left(\frac{\sqrt{3}}{2}, \frac{3}{2}, -1\right)$. Now we just need to normalize:

$\|\vec{B}\| = \sqrt{\vec{B}\cdot\vec{B}} = 2$, and therefore: $\frac{1}{\|\vec{B}\|}\vec{B} = \hat{B} = \left(\frac{\sqrt{3}}{4}, \frac{3}{4}, -\frac{1}{2}\right)$. Note, the negative: $-\hat{B}$ is also acceptable.

(2 points)

b) Find the point (x_0, y_0, z_0) such that the vectors $\vec{A} = (-1, 0, 8)$ and $\vec{B} = (0, -4, -16)$ are simultaneously tangent to the surface given by $x^2 - y^2 + z = 1$

Solution:

$\vec{A}$ and $\vec{B}$ are tangent and therefore orthogonal to the normal: $\vec{n} = \nabla\left(x^2 - y^2 + z\right) \mid_{(x_0,y_0,z_0)} = (2x_0, -2y_0, 1)$.

(3 points) Orthogonality implies the relations:

$\vec{A}\cdot\vec{n} = 0$ and $\vec{B}\cdot\vec{n} = 0$, writing it out: $\vec{A}\cdot\vec{n} = -2x_0 + 8 = 0 \Rightarrow x_0 = 4$ and $\vec{B}\cdot\vec{n} = 8y_0 - 16 = 0 \Rightarrow y_0 = 2$.

Now use the relation $x_0^2 - y_0^2 + z_0 = 1$ to find $z_0 = -11$, and we have: $(x_0, y_0, z_0) = (4, 2, -11)$ (2 points)

Problem 2 (10 points)

a) During lecture we saw the double product rule, $\left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial y}{\partial x}\right) = 1$, and triple product rule $\left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial y}{\partial z}\right)\left(\frac{\partial z}{\partial x}\right) = -1$.

Use the contour,

$f(x, y, z, t) = \frac{x\,y\,z}{\ln(t)} = 1$, to guess what the quadruple product rule is $\left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial y}{\partial z}\right)\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial t}{\partial x}\right) = ?$ That is, compute the four different derivatives and take the product (for sake of clarity write each expression resulting from a derivative only in terms of variables x, y, z, and be sure to not change variables until after computing the derivatives)

Solution:

Take the relation $f = 1$, and solve for each variable:

$x = \frac{1}{y\,z}\ln(t)$, $y = \frac{1}{x\,z}\ln(t)$, $z = \frac{1}{x\,y}\ln(t)$, $t = e^{x\,y\,z}$, and then take derivatives:

$\frac{\partial x}{\partial y} = -\frac{1}{y^2\,z}\ln(t)$, $\frac{\partial y}{\partial z} = -\frac{1}{z^2\,x}\ln(t)$, $\frac{\partial z}{\partial t} = \frac{1}{x\,y\,t}$, $\frac{\partial t}{\partial x} = y\,z\,e^{x\,y\,z}$, let's re-write them without the t:

$\frac{\partial x}{\partial y} = -\frac{x}{y}$, $\frac{\partial y}{\partial z} = -\frac{y}{z}$, $\frac{\partial z}{\partial t} = \frac{1}{x\,y}e^{-x\,y\,z}$, $\frac{\partial t}{\partial x} = y\,z\,e^{x\,y\,z}$, and now take the product and everything cancels out:

$\left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial y}{\partial z}\right)\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial t}{\partial x}\right) = 1$ (7 points)

b) Use the contour of the n-variable function $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i = 1$, to determine the n-tuple product rule, $\prod_{i=1}^{n}\left(\frac{\partial x_i}{\partial x_{i+1}}\right) = ?$

(Note we will interpret a subscript of n+1 as 1 in the product formula as in "clock arithmetic", recall $\prod$ means product and $\sum$ means sum, hint: don't overthink it)

Solution:

Take the relation $f = 1$, and solve for each variable, let's solve for the generic $x_i = 1 - \sum_{j \neq i} x_j$. Notice that x_{i+1} appears on the right as a lonely monomial, so the derivative is easy: $\frac{\partial x_i}{\partial x_{i+1}} = -1$ for any i. Therefore the product rule:

$\prod_{i=1}^{n}\left(\frac{\partial x_i}{\partial x_{i+1}}\right) = (-1)^n$ (3 points)

Notice that this reproduces the double product rule (n=2): $\left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial y}{\partial x}\right) = (-1)^2 = 1$

And the triple product rule (n=3): $\left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial y}{\partial z}\right)\left(\frac{\partial z}{\partial x}\right) = (-1)^3 = -1$

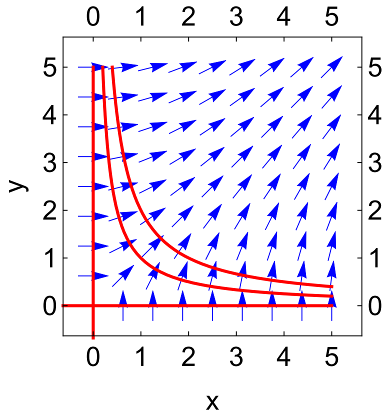
And part a) the quadruple product rule (n=4): $\left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial y}{\partial z}\right)\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial t}{\partial x}\right) = (-1)^4 = 1$

Problem 3 (10 points)

a) Together, in the same *positive* quadrant of the $x\,y$ plane, *very roughly* sketch the contours of $f(x, y) = x\,y$ for $f = 0, 1, 2$, and without doing any calculations *very roughly* sketch the vector field $\vec{\nabla} f$ (draw about ~6-9 vectors relatively spaced out). (You may do some calculations if you need to aid yourself).

Solution:

Something vaguely resembling the following will suffice. Note: it should have been easy enough to draw arrows that are orthogonal to the contours. (6 points)



b) Find a direction, i.e., a unit vector, such that the directional derivative of $f(x, y) = x\,y$, at the point $(x_0, y_0) = (1, 1)$ is exactly half of its max value $\left(i.e., \ \frac{1}{\sqrt{2}}\right)$.

(Hint: any unit vector can be written as $\hat{n} = \left(p, \sqrt{1-p^2}\right)$ for $0 \le p \le 1$, also recall the quadratic formula:

$a\,p^2 + b\,p + c = 0 \Rightarrow p = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}$)

Solution:

Recall the directional derivative has the form: $\vec{\nabla} f \cdot \hat{n} = \|\vec{\nabla} f\| \ \|\hat{n}\| \cos\theta = \|\vec{\nabla} f\| \cos\theta$, which of course has a max at $\theta = 0$,

so the max is $\|\vec{\nabla} f\|$ (but we already knew that since the gradient points in the direction of greatest increase). Now we want to find a unit vector that satisfies: $\vec{\nabla} f \cdot \hat{n} = \frac{1}{2} \|\vec{\nabla} f\|$

The gradient is $\vec{\nabla} f = (y, x) \,|_{(x_0, y_0) = (1,1)} = (1, 1)$, and let's use $\hat{n} = \left(p, \sqrt{1-p^2}\right)$, so that the directional derivative relation takes the form:

$\vec{\nabla} f \cdot \hat{n} = \frac{1}{2} \|\vec{\nabla} f\| \Rightarrow (1, 1) \cdot \left(p, \sqrt{1-p^2}\right) = p + \sqrt{1-p^2} = \frac{1}{2}\sqrt{2} = \frac{1}{\sqrt{2}}$, (2 points) so all we need to do is find p. Isolate the square root:

$\sqrt{1-p^2} = \frac{1}{\sqrt{2}} - p$, then square both sides:

$1 - p^2 = \frac{1}{2} - \sqrt{2}\,p + p^2$

$\Rightarrow \ 2\,p^2 - \sqrt{2}\,p - \frac{1}{2} = 0$

$\Rightarrow p = \frac{1}{4}\left(\sqrt{2} \pm \sqrt{6}\right) = \pm\sqrt{\frac{1}{16}\left(\sqrt{2} \pm \sqrt{6}\right)^2} = \pm\sqrt{\frac{1}{2} \pm \frac{\sqrt{3}}{4}} = \pm\sqrt{1-p^2}$

(2 points)

If you found a solution for p, you may well deserve full credit. To write down an answer explicitly, we have two option: $\hat{n} = \left(-\sqrt{\frac{1}{2} - \frac{\sqrt{3}}{4}}, \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{4}}\right) = \frac{1}{4}\left(\sqrt{2} - \sqrt{6}, \sqrt{2} + \sqrt{6}\right)$, or,

$\hat{n} = \left(\sqrt{\frac{1}{2} + \frac{\sqrt{3}}{4}}, -\sqrt{\frac{1}{2} - \frac{\sqrt{3}}{4}}\right) = \frac{1}{4}\left(\sqrt{2} + \sqrt{6}, \sqrt{2} - \sqrt{6}\right)$

Problem 4 (10 points)

For each part of this problem, write all answers in the form, $a\,x + b\,y + c = 0$

a) Using the 1st order Taylor series, compute an expression for the tangent line of the graph of y = ln(x), at the point $(x_0, y_0) = (1, 0)$

Solution:

$y = \ln(x) \simeq \ln(x_0) + \frac{1}{x_0}(x - x_0) = x - 1 \Rightarrow -x + y + 1 = 0$ (3 points)

b) Use the following parametrization of the above graph, $\vec{r}(t) = (e^t, t)$, to find an expression for the tangent line at the point $(x_0, y_0) = (1, 0)$

Solution:

$\vec{r}(t_0) = (e^{t_0}, t_0) = (x_0, y_0) = (1, 0) \Rightarrow t_0 = 0,\ \vec{r}\,'(t_0) = (1, 1) \Rightarrow \vec{n} = (-1, 1)$

$\Rightarrow \vec{n} \cdot (x - x_0, y - y_0) = 0 \Rightarrow -x + y + 1 = 0$ (3 points)

c) Use the gradient of the function $g(x, y) = y - \ln(x)$, to find an expression for the tangent line to the contour $g = 0$, at the point $(x_0, y_0) = (1, 0)$

Solution:

$\vec{n} = \nabla g\,|_{(x_0, y_0)} = \left(-\frac{1}{x_0}, 1\right)|_{(x_0, y_0)} = (-1, 1)$

$\Rightarrow \vec{n} \cdot (x - x_0, y - y_0) = 0 \Rightarrow -x + y + 1 = 0$ (3 points)

Notice that all of the answers are the same (1 point)