MATH 20C - Allen - Midterm 1

Show all work. No credit might be given for unsupported answers, even if correct.

Note the magenta color is only a suggestion for the partial credit breakdown. Alternative or complex solutions must necessarily take a more personalized approach

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Problem 0 (1 point)

Write your name, PID, and section number on the front of your bluebook.

Solution:

Professor, 53014497, C00 (1 point)

Problem 1 (10 points)

Alice and Bob are each confined to two separate but parallel planes. The planes can be given implicitly as:

plane 1: $x - 2y + z + 2 = 0$

plane 2: $x - 2y + z - 1 = 0$

Alice's initial position is given by $\vec{A} = (1, 1, 2)$, while Bob's initial position is given by $\vec{B} = (-1, 2, 3)$.

a) Which plane is Alice on? Which plane is Bob on?

b) What is the distance between Alice and Bob initially?

c) If Alice and Bob are able to move around freely within their respective planes, what is the smallest separation distance they can achieve?

Solution:

a) Alice lives on plane 2: $(1) - 2(1) + (2) - 1 = 0$

Bob lives on plane 1: $(-1) - 2(2) + (3) - 1 = 0$ (3 points)

b) $\|\vec{A} - \vec{B}\| = \sqrt{(2, -1, -1) \cdot (2, -1, -1)} = \sqrt{6}$ (3 points)

c) $\|\text{proj}_{\vec{n}}(\vec{A} - \vec{B})\| = \left\|\left(\frac{\vec{n} \cdot (\vec{A} - \vec{B})}{\vec{n} \cdot \vec{n}}\right)\vec{n}\right\| = \left\| \frac{(1,-2,1)\cdot(2,-1,-1)}{(1,-2,1)\cdot(1,-2,1)} (1, -2, 1) \right\| = \sqrt{\frac{3}{2}}$ (4 points)

Problem 2 (10 points)

A plane passes through the following vectors, $\vec{A} = (3, -4, 4)$, $\vec{B} = (-3, -4, 2)$, $\vec{C} = (4, -3, 4)$.
Find an implicit form for the plane of the form, $ax + by + cz + d = 0$, using whichever method you like (an algebraic elimination or a geometric construction).

Solution:

Method 1 (algebraic elimination):

Here's a parametrization: $\vec{r}(s, t) = (\vec{A} - \vec{B})\,t + (\vec{A} - \vec{C})\,s + \vec{A}$, which gives:

$x = 6t - s + 3$ (4 points)
$y = -s - 4$
$z = 2t + 4$

We can immediately solve for s and t with the last two equations, and substitute into the first equation:

$y = -s - 4 \Rightarrow s = -y - 4$ (3 points)
$z = 2t + 4 \Rightarrow t = \frac{1}{2}z - 2$

$$\left.\begin{array}{l} y = -s - 4 \Rightarrow s = -y - 4 \\ z = 2t + 4 \Rightarrow t = \frac{1}{2}z - 2 \end{array}\right\} \Rightarrow x = 6t - s + 3 = 6\left(\frac{1}{2}z - 2\right) - (-y - 4) + 3$$

After simplifying, we get:

$x - y - 3z + 5 = 0$ (3 points)

Method 2 (geometric construction):

Let $\vec{\Delta}_1 = \vec{A} - \vec{B} = (6, 0, 2)$, and, $\vec{\Delta}_2 = \vec{A} - \vec{C} = (-1, -1, 0)$, be two vectors that lie along the plane.
Then a normal vector can be given by the cross product

$\vec{n} = \vec{\Delta}_1 \times \vec{\Delta}_2 = (2, -2, -6)$ (4 points)

Let's let $\vec{r}_0 = \vec{A}$ (3 points) and then we can immediately write down the answer:

$\vec{n} \cdot (\vec{r} - \vec{r}_0) = (2, -2, -6) \cdot (x - 3, y + 4, z - 4) = 2x - 2y - 6z + 10 = 0$

We can optionally factor out a 2:

$2(x - y - 3z + 5) = 0$ (3 points)

Problem 3 (10 points)

Find a parametrization of the line that is the intersection of the following planes:
plane 1: $2x + y - z + 2 = 0$
plane 2: $3x - y + 2z - 1 = 0$

You may write your answer in the form:
$x(t) = \ldots,\ y(t) = \ldots,\ z(t) = \ldots$

Solution:

A parametrization of a line has the general form:

$\vec{r}(t) = (\text{Any vector lying along the line})\,t + (\text{any vector that points to the line})$

A vector that lies along that line, also lies along *both* planes, and therefore is orthogonal to each plane's normal. A vector orthogonal to two others can be found through the cross product:

$\vec{\Delta} = \vec{n}_1 \times \vec{n}_2 = (2, 1, -1) \times (3, -1, 2) = (1, -7, -5)$ (4 points)

Now we just need to find one single point on the line. All points on the line must satisfy the two implicit equations simultaneously.
A line generally will cross all coordinate planes (xy-plane, xz-plane, and yz-plane), so we can use this fact to simplify find a solution: let $x = 0$:

$2(0) + y - z + 2 = 0$
$3(0) - y + 2z - 1 = 0$
If you add the equations together, we get $z = -1$, and back-substituting this value gives $y = -3$:
$\vec{r}_0 = (0, -3, -1)$ (4 points)

Now fill in the blanks:
$\vec{r}(t) = \vec{\Delta} t + \vec{r}_0 = (1, -7, -5)\, t + (0, -3, -1)$, and so we have the coordinates as functions of a parameter:
$x(t) = t$ (2 points)
$y(t) = -7t - 3$
$z(t) = -5t - 1$
As always, for parametrizations, there are an infinite number of correct answers

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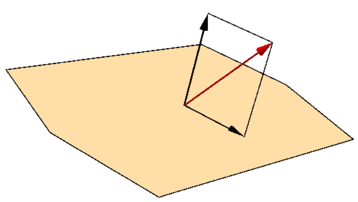
Problem 4 (10 points)

Find the projection of the vector $\vec{A} = (-2, 3, 4)$ onto a *plane*.
Let the plane pass through the origin and be given implicitly by $x + 3y + 3z = 0$.

In other words, find a vector that lies in the plane whose difference with $\vec{A}$ is normal to the plane, as shown in the figure.

Solution:

A picture might help:



The red vector is $\vec{A}$. The perpendicular vector is the projection of $\vec{A}$ onto the normal $\vec{n}$. The difference gives the answer:

$$\text{proj}_{\text{plane}} \vec{A} = \vec{A} - \text{proj}_{\vec{n}} \vec{A} = \vec{A} - \left(\frac{\vec{n} \cdot \vec{A}}{\vec{n} . \vec{n}}\right) \vec{n} = (-2, 3, 4) - (1, 3, 3) = (-3, 0, 1)$$

(5 points for $\text{proj}_{\vec{n}} \vec{A}$, 5 points for correct vector difference)

Notice that this vector satisfies the plane equation: $(-3) + 3(0) + 3(1) = 0$.

Also, notice that in this problem, the numbers worked in your favor so that: $\text{proj}_{\vec{n}} \vec{A} = \vec{n}$, i.e., it's possible to have accidentally gotten the right answer!