Problem 1 (10 points)

Evaluate the quadruple integral
\[ \int_0^2 \int_0^1 \int_0^{x+y} \int_y^{x+y} \, dt \, dz \, dy \, dx \]

Solution
\[ \int_0^2 \int_0^1 \int_0^{x+y} \int_y^{x+y} \, dt \, dz \, dy \, dx = \int_0^2 \int_0^{x^2 - y^2} (x-y) \, dy \, dx = \int_0^2 (x^2 - \frac{1}{3}) \, dx = 2 \]

Integrate \([1, \{x, 0, 2\}, \{y, 0, 1\}, \{z, x, x+y\}, \{t, y, x\}]\)

Problem 2 (10 points)

The area of a region is given by a sum of double integrals:
\[ \text{Area} = \int_0^1 \int_0^{\sqrt{1-x^2}} \, dy \, dx + \int_1^2 \int_0^{\sqrt{1-x^2}} \, dy \, dx \]

Simplify the expression for area by reversing the order of integration and finding the appropriate limits:
\[ \text{Area} = \int_{y_0}^{y_1} \int_{x_0(y)}^{x_1(y)} \, dx \, dy \]

You do not need to evaluate the integral. Hint: the inner curve is given by \(x^2 + y^2 = 1\), while the outer curve is given by \(\frac{x^2}{2} + y^2 = 1\)

Solution

CylindricalDecomposition \([x^2 + y^2 > 1 \&\& \frac{x^2}{2} + y^2 < 1 \&\& x > 0 \&\& y > 0, \{y, x\}]\)

\[ 0 < y < 1 \&\& \sqrt{1-y^2} < x < \sqrt{2 - 2y^2} \]

Area = \[ \int_0^1 \int_{\sqrt{1-y^2}}^{\sqrt{2-2y^2}} \, dx \, dy \]
Just for fun (not necessary to receive full credit):

\[ \text{Integrate}\left[1, \{y, 0, 1\}, \left\{x, \sqrt{1-y^2}, \sqrt{2-2y^2}\right\}\right] \]

\[ \frac{1}{4} \left(-1 + \sqrt{2}\right) \pi \]

1/4 the area of an ellipse (π a b, with a/b the semi-major/minor axes) minus 1/4 the area of a disk (π r^2):

\[ \frac{1}{4} \pi \sqrt{2} - \frac{1}{4} \pi \]

Problem 3 (10 points)

Find the point \((x_0, y_0)\) such that the following function is locally extremized:

\[ f(x, y) = x^2 + 4xy - 10x - y^2 + 5, \]

and determine whether it corresponds to a max, min, or saddle point.

Solution

\[ f = \frac{1}{2} (x - 1, y - 2). \left( \begin{array}{c} 2 \\ 4 \end{array} \right) \cdot (x - 1, y - 2) \] // Expand

\[ \text{Solve}\left[\left\{\partial_x f, \partial_y f\right\} = (0, 0)\right]\]

\[ 5 - 10x + x^2 + 4xy - y^2 \]

\[ \{x \rightarrow 1, y \rightarrow 2\} \]

\[ H = \text{Table}[\partial_1 \partial_2 f, \{i, \{x, y\}\}, \{j, \{x, y\}\}]; \]

\[ H \] // \text{MatrixForm}

\[ \begin{pmatrix} 2 & 4 \\ 4 & -2 \end{pmatrix} \]

\[ H \] // \text{Det}

\[ -20 \]

Saddle point at \((x_0, y_0) = (1, 2)\)

Problem 4 (10 points)

Write an expression of the form, \(ax + by + c = 0\), for the tangent line at the point \((x_0, y_0) = (2, 1)\) for the implicit curve:

\[ x^3 - \frac{9xy}{2} + y^3 = 0 \]
Solution

Take the gradient of the constraint and dot into \((x - x_0, y - y_0)\):

\[
f = x^3 + y^3 - \frac{9}{2} xy;
\]

\[
n = \{\partial_x f, \partial_y f\} \cdot (x \to 2, y \to 1); \\
n \cdot (x - 2, y - 1) \quad \text{// Expand} \\
-9 + \frac{15x}{2} - 6y
\]

Problem 5 (10 points)

A path you might trace out along a carousel can be parametrized as follows:

\[
\vec{r}(t) = (2 \cos(2 \pi t), 2 \sin(2 \pi t), \sin(10 \pi t)).
\]

Find parametric equations for the tangent line, \(\vec{T}(t)\), at the point \(t_0 = 1/4\)

Solution

\[
\text{Clear}[r]; \\
t0 = 1/4; \\
r[t_] = \{2 \cos[2 \pi t], 2 \sin[2 \pi t], \sin[10 \pi t]\}; \\
T[t_] = r[t0] + t r'[t0] \\
\{-4 \pi t, 2, 1\}
\]

Problem 6 (10 points)

Evaluate the following derivatives,

\[
\frac{\partial f}{\partial u} \quad \text{and} \quad \frac{\partial f}{\partial v},
\]

at \((x_0, y_0, z_0) = (1,0,2)\), and \((u_0, v_0) = (0, -1)\), given the following data, \(f(x, y, z) = x^2 + yz\), and \(x(u, v) = u + 1, \quad y(u, v) = u v, \quad z(u,v) = -v+1.\)
Solution

\( f = x^2 + yz; \)
\( r = \{u + 1, uv, -v + 1\}; \)
\( \text{gradf} = \{\partial_z f, \partial_y f, \partial_x f\} \)/. \{x \rightarrow 1, y \rightarrow 0, z \rightarrow 2\}
\( \text{drdu} = \partial_u r \)/. \{u \rightarrow 0, v \rightarrow -1\}
\( \text{drdv} = \partial_v r \)/. \{u \rightarrow 0, v \rightarrow -1\}
\( \{2, 2, 0\} \)
\( \{1, -1, 0\} \)
\( \{0, 0, -1\} \)
\( \text{gradf}.\text{drdu} \)
\( \text{gradf}.\text{drdv} \)
\( 0 \)
\( 0 \)

Problem 7 (10 points)

Write an expression of the form, a \( x + b y + c z + d = 0 \), for the tangent plane at the point \( (x_0, y_0, z_0) = (1, 1, 1) \), for the following implicit surface:

\( f(x, y, z) = xyz = 1 \)

Solution

The point lives on the surface:

\( \{x_0, y_0, z_0\} = \{1, 1, 1\}; \)
\( xyz \)/. \{x \rightarrow x0, y \rightarrow y0, z \rightarrow z0\}
\( 1 \)

\( f = xyz; \)
\( n = \{\partial_z f, \partial_y f, \partial_x f\} \)/. \{x \rightarrow x0, y \rightarrow y0, z \rightarrow z0\};
\( n.\{x - x0, y - y0, z - z0\} \)/ Expand
\( -3 + x + y + z \)

Problem 8 (10 points)

Find the extrema of \( f \), given the constraint \( g \):

\( f(x, y) = -x^2 - 2xy + x - y^2 + 3y, \ g(x, y) = y - x = -1 \)
Solution

\[ f = -x^2 - 2xy + x - y^2 + 3y; \]
\[ g = y - x; \]
\[ \{\partial_x f = \lambda \partial_x g, \partial_y f = \lambda \partial_y g, \, g = -1\} \]
\[ \text{Solve[\{\{x, y, \lambda\}\}]} \]
\[ \{1 - 2x - 2y = -\lambda, \, 3 - 2x - 2y = \lambda, \, -x + y = -1\} \]
\[ \{(x \to 1, \, y \to 0, \, \lambda \to 1)\} \]

We only care about \((x_0, y_0)\) so \((x_0, y_0) = (1, 0)\)

Problem 9 (10 points)

Find all two extrema of \(f\), given two constraints \(g_1\) and \(g_2\):
\[ f(x,y) = z \]
\[ g_1(x, y) = xy = 1 \]
\[ g_2(x, y) = x + y + z = 4 \]

Solution

\[ f = z; \]
\[ g_1 = xy; \]
\[ g_2 = x + y + z; \]
\[ \{\partial_x f = \lambda_1 \partial_x g_1 + \lambda_2 \partial_x g_2, \partial_y f = \lambda_1 \partial_y g_1 + \lambda_2 \partial_y g_2, \]
\[ \partial_z f = \lambda_1 \partial_z g_1 + \lambda_2 \partial_z g_2, \, g_1 = 1, \, g_2 = 4\} \]
\[ \text{Solve[\{\{x, \, y, \, z, \, \lambda_1, \, \lambda_2\}\}]} \]
\[ \{0 = y \lambda_1 + \lambda_2, \, 0 = x \lambda_1 + \lambda_2, \, 1 = \lambda_2, \, x y = 1, \, x + y + z = 4\} \]
\[ \{(x \to -1, \, y \to -1, \, z \to 6, \, \lambda_1 \to 1, \, \lambda_2 \to 1), \, (x \to 1, \, y \to 1, \, z \to 2, \, \lambda_1 \to -1, \, \lambda_2 \to 1)\} \]

We only care about \((x_0, y_0, z_0)\) so \((x_0, y_0, z_0) = (-1, -1, 6), (1, 1, 2)\)