MATH 20C - Allen - Final

Show all work. No credit might be given for unsupported answers, even if correct.

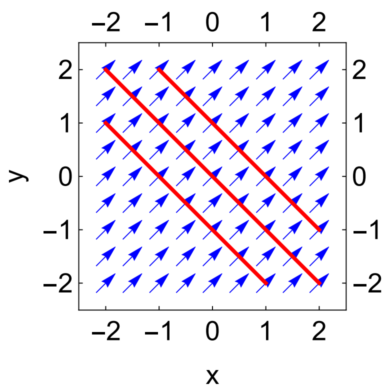
Problem 0 (1 point)

Write your name, PID, and section number on the front of your bluebook.

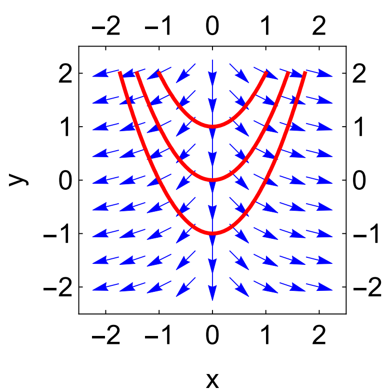
Problem 1 (10 points)

For each part of this problem, write your answer on a single plot (so 3 plots in total): very roughly sketch the contours of the following functions, and without doing any calculations, sketch also the gradient, $\vec{\nabla} f$, (draw perhaps 6-9 arrows, not necessarily to scale).

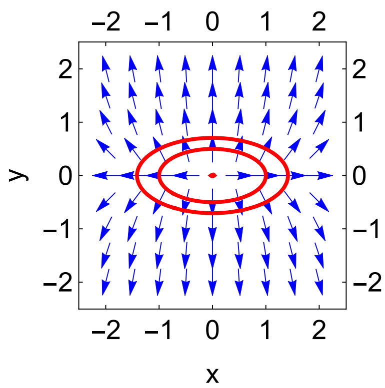
a) $f(x, y) = x + y$, for $f = -1, 0, 1$



b) $f(x, y) = x^2 - y$, for $f = -1, 0, 1$



b) $f(x, y) = x^2 + 4 y^2$, for $f = 0, 1, 2$



(2 points correct contours, 1 point gradient orthogonal to contours)
(1 point for remembering that the gradient should point towards the contours of larger value)

Problem 2 (10 points)

For each part of this problem, write all answers in the form, $a\,x + b\,y + c\,z + d = 0$, for the tangent plane at the point $(x_0, y_0, z_0) = \left(\frac{1}{2}, \frac{1}{3}, \sqrt{2}\right)$.

a) Using the 1st order Taylor series, compute an expression for the tangent plane of the graph of $z = f(x, y) = \sqrt{4\,x^2 + 9\,y^2}$.

$z = f(x, y) = \sqrt{4\,x^2 + 9\,y^2} \simeq f(x_0, y_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) = \sqrt{2} + \sqrt{2}\left(x - \frac{1}{2}\right) + \frac{3}{\sqrt{2}}\left(y - \frac{1}{3}\right),$

now rearrange to get:

$\sqrt{2}\,x + \frac{3}{\sqrt{2}}\,y - z = 0,$

optionally one could multiply the above by $\sqrt{2}$ to get:

$2\,x + 3\,y - \sqrt{2}\,z = 0$ (3 points)

b) Use the parametrization of the above graph, $\vec{r}(s, t) = \left(\frac{s}{2}\sin t,\ \frac{s}{3}\cos t,\ s\right)$, to find an expression for the tangent plane.

First note that: $\vec{r}(s_0, t_0) = (x_0, y_0, z_0) \Rightarrow (s_0, t_0) = \left(\sqrt{2}, \frac{\pi}{4}\right)$. The tangents are :

$\frac{\partial \vec{r}}{\partial s} = \left(\frac{1}{2\sqrt{2}}, \frac{1}{3\sqrt{2}}, 1\right)$, and $\frac{\partial \vec{r}}{\partial t} = \left(\frac{1}{2}, -\frac{1}{3}, 0\right)$, which gives the normal $\vec{n} = \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} = \left(\frac{1}{3}, \frac{1}{2}, -\frac{1}{3\sqrt{2}}\right)$, and we write the answer down:

$\vec{n}\cdot(x - x_0, y - y_0, z - z_0) = \left(\frac{1}{3}, \frac{1}{2}, -\frac{1}{3\sqrt{2}}\right)\cdot\left(x - \frac{1}{2}, y - \frac{1}{3}, z - \sqrt{2}\right) = \frac{1}{3}\,x + \frac{1}{2}\,y - \frac{1}{3\sqrt{2}}\,z = 0,$

optionally one could multiply the above by 6 to get:

$2\,x + 3\,y - \sqrt{2}\,z = 0$ (3 points)

c) Use the gradient of the function $g(x, y, z) = z^2 - 4\,x^2 - 9\,y^2$, to find an expression for the tangent plane to the contour $g = 0$.

The gradient gives the normal vector immediately: $\vec{n} = \vec{\nabla} g = \left(-4, -6, 2\sqrt{2}\right)$, so we write down the answer as:

$\vec{n}\cdot(x-x_0,\ y-y_0,\ z-z_0) = \left(-4,\ -6,\ 2\sqrt{2}\right)\cdot\left(x-\frac{1}{2},\ y-\frac{1}{3},\ z-\sqrt{2}\right) = -4x-6y+2\sqrt{2}\,z=0,$

optionally one could multiply the above by $-\frac{1}{2}$ to get:

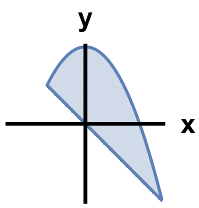
$2x+3y-\sqrt{2}\,z=0$ (3 points)

(1 point for *not* giving up here and submitting a blank exam)w

---

Problem 3 (10 points)

The region shown below is bounded by curves $y=-x$ and $y=-x^2+2$.

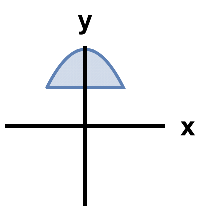


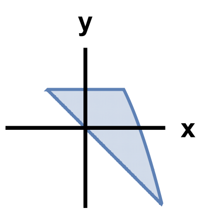
a) Write down a double integral expression for the area, with y as the inner integral and x as the outer integral. Evaluate the integral.

First we should we where the curves intersect:
$y=-x = -x^2+2 \Rightarrow x^2-x-2 = (x-2)(x+1) = 0 \Rightarrow (x,y) = (-1,1),\ (2,-2)$
The lower and upper curves for y should be simply enough, so we get:
$\int_{-1}^{2}\int_{-x}^{-x^2+2} dy\,dx = \int_{-1}^{2}\left(-x^2+2+x\right)dx = -\frac{x^3}{3}+2x+\frac{x^2}{2}\ \Big|_{-1}^{2} = -\frac{8}{3}+4+2-\frac{1}{3}+2-\frac{1}{2} = -3+8-\frac{1}{2} = \frac{9}{2}$
(4 points correct limits, 1 point correct answer)

b) Write down a double integral expression for the area, with x as the inner integral and y as the outer integral (you will need two double integrals). Evaluate the integrals.

The piecewise nature of the lower bound for x suggests that we break the region into two pieces:





So we write down a double integral for each piece:

$$\int_1^2\int_{-\sqrt{2-y}}^{+\sqrt{2-y}} dx\,dy + \int_{-2}^1\int_{-y}^{+\sqrt{2-y}} dx\,dy$$

$$= \int_1^2 2\sqrt{2-y}\,dx + \int_{-2}^1\left(\sqrt{2-y} + y\right)dx$$

$$= -\frac{4}{3}(2-y)^{3/2}\,\Big|_1^2 - \frac{2}{3}(2-y)^{3/2}\,\Big|_{-2}^1 + \frac{y^2}{2}\,\Big|_{-2}^1 = \frac{4}{3} - \frac{2}{3} + \frac{16}{3} + \frac{1}{2} - 2$$

$$= 6 - 2 + \frac{1}{2}$$

$$= \frac{9}{2}$$

(4 points correct limits, 1 point correct answer)

Problem 4 (10 points)

Find both the absolute max and absolute min (i.e., the coordinates and function value) within the unit disk of radius root two, i.e., $g(x, y) = x^2 + y^2 \leq 2$, of the function $f(x, y) = x^2 - 2x + y^2 + 1$. (Hint: there are 3 possible extrema)

Look for unconstrained extrema on the interior of the disk:

$\vec{\nabla} f = (2x - 2,\ 2y) = (0, 0) \Rightarrow (x, y) = (1, 0)$

Look for constrained extrema on the boundary of the disk:

$\vec{\nabla} f = (2x - 2,\ 2y) = \lambda \vec{\nabla} g = \lambda(2x, 2y)$, so we have 3 equations and 3 unknowns:

$2x - 2 = \lambda 2x$,
$2y = \lambda 2y$,
$x^2 + y^2 = 1$,

looking at the 2nd equation, let's assume $y \neq 0$, so we can divide it out $\Rightarrow \lambda=1$, but then the first equation becomes $-2 = 0$, so our assumption that $y \neq 0$, is not valid, and therefore $y = 0$. In this case, we can use the 3rd equation to find that $x = \pm\sqrt{2}$.

So we have two constrained extrema: $(x, y) = \left(+\sqrt{2}, 0\right), \left(-\sqrt{2}, 0\right)$.

Evaluating the function at each extrema:

$f\left(\sqrt{2}, 0\right) = \left(\sqrt{2} - 1\right)^2$

$f\left(-\sqrt{2}, 0\right) = \left(-\sqrt{2} - 1\right)^2$

$f(1, 0) = 0$

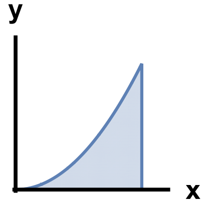
The absolute max must be at $(x, y) = \left(-\sqrt{2}, 0\right)$, and the absolute min must be at $(x, y) = (1, 0)$.

(8 points correct extrema, 2 points correct conclusion about max/min)

Problem 5 (10 points)

Evaluate the following integral by reversing the order of integration: $\int_0^1\int_{\sqrt{y}}^1 \sqrt{x^3 + 1}\,dx\,dy$

First you need to figure out the region: $x_l = \sqrt{y}$ and $x_u = 1$, so we have the following:



Now we just rewrite the integral with the order reversed (the original integrand does not change!) so we get:

$\int_0^1 \int_0^{x^2} \sqrt{x^3+1}\, dy\, dx = \int_0^1 x^2 \sqrt{x^3+1}\, dx$, use substitution: $u = x^3 + 1$, $du = 3x^2\, dx$

$= \int \frac{du}{3} \sqrt{u} = \frac{2}{9} u^{3/2} = \frac{2}{9} (x^3+1)^{3/2} \,|_0^1 = \frac{2}{9}\left(2\sqrt{2} - 1\right)$

(7 points correct limits, 2 points substitution, 1 point correct answer)

Problem 6 (10 points)

Find the coordinates of the two extrema of f, given two constraints $g_1 = 1$ and $g_2 = 0$:

$f(x, y, z) = 2x^2 y + z^2$
$g_1(x, y, z) = x + y = 1$
$g_2(x, y, z) = 2x - z = 0$

Plug and chug:

$\vec{\nabla} f = \lambda_1 \vec{\nabla} g_1 + \lambda_2 \vec{\nabla} g_2$, gives 5 equations and 5 unknowns:

$4xy = \lambda_1 + 2\lambda_2$,
$2x^2 = \lambda_1$,
$2z = \lambda_2$,
$x + y = 1$,
$2x - z = 0$,

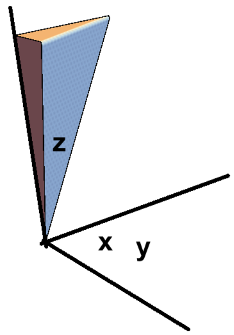
Add combinations of the first 3 equations to cancel the lambda's on the right:
$4xy - 2x^2 - 4z = 0$, and use the last 2 equations to eliminate $y = 1 - x$ and $z = 2x$:
$4x(1-x) - 2x^2 - 8x = 4x - 4x^2 - 2x^2 - 8x = x(12 - 6x) = 0$, $\Rightarrow x = 0, 2$, which can be used to find y and z:

Therefore the two extrema are at: $(x, y, z) = (0, 1, 0)$, $(2, -1, 4)$

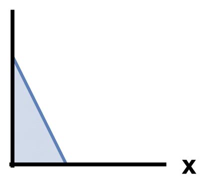
(7 points for one extrema, 3 points for another)

Problem 7 (10 points)

The 3-D region show below is confined to the positive octant and bounded by two surfaces: $4x + 2y - z = 0$ and $z = 1$. Compute the volume of the region with a triple integral.



Remember, it helps to look at the shadow of the volume onto the lower dimensions. The shadow on the xy-plane would look like:



which of course comes from the top surface $z = 1$. So at $z = 1$, the largest value of x is $\frac{1}{4}$. At $z = 1$, the upper bound on y is $\frac{1-4x}{2}$

So writing down the integral:

$$\int_0^{1/4}\int_0^{1/2-2x}\int_{4x+2y}^{1} dz\,dy\,dx$$
$$= \int_0^{1/4}\int_0^{1/2-2x}(1-4x-2y)\,dy\,dx$$
$$= \int_0^{1/4}(y-4xy-y^2)\Big|_0^{1/2-2x}\,dx$$
$$= \int_0^{1/4}\left(\left(\tfrac{1}{2}-2x\right)-4x\left(\tfrac{1}{2}-2x\right)-\left(\tfrac{1}{2}-2x\right)^2\right)dx$$
$$= \int_0^{1/4}\left(\tfrac{1}{2}-2x-4x\left(\tfrac{1}{2}-2x\right)-\left(\tfrac{1}{2}-2x\right)^2\right)dx$$
$$= \int_0^{1/4}\left(\tfrac{1}{2}-2x-2x+8x^2-\tfrac{1}{4}+2x-4x^2\right)dx$$
$$= \int_0^{1/4}\left(\tfrac{1}{4}-2x+4x^2\right)dx$$
$$= \left(\tfrac{1}{4}x-x^2+\tfrac{4}{3}x^3\right)\Big|_0^{1/4}$$
$$= \left(\tfrac{1}{16}-\tfrac{1}{16}+\tfrac{1}{3}\tfrac{1}{16}\right)$$
$$= \left(\tfrac{1}{48}\right)$$

(8 points correct limits, 2 points for good effort towards the final answer)

Problem 8 (10 points)

The function, $f(x, y, z) = -2x^2 + 4xy - 5y^2 + 6y - 3z^2 - 3$, has one extrema. Locate it and use the 2nd derivative test to determine whether the extrema is a max or min.

Unconstrained extrema have vanishing gradient:

$\vec{\nabla} f = \vec{0}$, which gives:

$(-4x+4y,\ 4x-10y+6,\ -6z) = (0, 0, 0)$,
The 3rd equation tells us $z = 0$, while the first equation says $x = y$, plug this into the 2nd equation:
$4x - 10x + 6 = -6x + 6 = 0 \Rightarrow x = 1$

So our extrema is $(x, y, z) = (1, 1, 0)$

The Hessian matrix is: $H = \begin{pmatrix} -4 & 4 & 0 \\ 4 & -10 & 0 \\ 0 & 0 & -6 \end{pmatrix}$, so by computing the determinants of submatrices:

$\det(-4) = -4 < 0$

$\det\begin{pmatrix} -4 & 4 \\ 4 & -10 \end{pmatrix} = 24 > 0$

$\det\begin{pmatrix} -4 & 4 & 0 \\ 4 & -10 & 0 \\ 0 & 0 & -6 \end{pmatrix} = -144 < 0$

So the signs go: -, +, -, and therefore the extrema corresponds to a max.

(5 points for extrema coordinates, 3 points for Hessian, 2 points for correct conclusion)

Problem 9 (10 points)

What is the angle between the two vectors:

1) the tangent to the curve $\vec{r}(t) = (t, t^2, t^3)$,

2) the normal to the implicit surface (pointing away from the origin), $f(x, y, z) = x + y + z - 3 = 0$, each at the point $(x_0, y_0, z_0) = (1, 1, 1)$. You may write your answer in terms of inverse trig.

The parametrized curve satisfies: $\vec{r}(t_0) = (x_0, y_0, z_0) \Rightarrow t_0 = 1$.

The first vector is a tangent to the curve, so $\frac{d\vec{r}}{dt} = (1, 2, 3)$.

The second vector is normal to the surface, which we can get with the gradient: $\vec{n} = \vec{\nabla} f = (1, 1, 1)$.
The angle between two vectors is given by:

$$\theta = \cos^{-1} \frac{\vec{\nabla} f \cdot \frac{d\vec{r}}{dt}}{\|\vec{\nabla} f\| \, \|\frac{d\vec{r}}{dt}\|} = \cos^{-1} \frac{(1,1,1)\cdot(1,2,3)}{\sqrt{1^2+1^2+1^2}\ \sqrt{1^2+2^2+3^2}} = \cos^{-1} \frac{6}{\sqrt{3}\ \sqrt{2}\ \sqrt{7}} = \cos^{-1} \sqrt{6/7}$$

(7 points for calculating the vectors correctly, 3 points for correct answer)

Problem 10 (10 points)

The temperature of a region is given by $T(x, y) = \sqrt{\frac{x}{y}}$

a) How does the temperature change at the point $(x_0, y_0) = (1, 2)$ in the direction $\hat{n} = \left(\frac{3}{5}, \frac{4}{5}\right)$?

The directional derivative is

$\vec{\nabla} T \cdot \hat{n} = \left(\frac{1}{2} x^{-1/2} y^{-1/2}, -\frac{1}{2} x^{1/2} y^{-3/2}\right) |_{(x,y)=(1,2)} \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = \left(\frac{1}{2\sqrt{2}}, -\frac{1}{4\sqrt{2}}\right) \cdot \left(\frac{3}{5}, \frac{4}{5}\right) = \frac{1}{10\sqrt{2}}$

b) How does the temperature change at the point $(x_0, y_0) = (1, 2)$ according to an observer walking along the path $\vec{r}(t) = (t^3, 2t^2)$?

The path satisfies $\vec{r}(t_0) = (x_0, y_0, z_0) \Rightarrow t_0 = 1$. The chain rule says

$\frac{dT}{dt} = \vec{\nabla} T \cdot \frac{d\vec{r}}{dt} = \left(\frac{1}{2\sqrt{2}}, -\frac{1}{4\sqrt{2}}\right) \cdot (3, 4) = \frac{1}{2\sqrt{2}}$

c) What is the ratio of answers to part b and a? What is the speed of the observer (i.e., the length of the tangent vector $\frac{d\vec{r}}{dr}$)?

part b/ part a = $\frac{1}{2\sqrt{2}} \Big/ \frac{1}{10\sqrt{2}} = 5$, $\quad || \frac{d\vec{r}}{dr} || = ||(3, 4)|| = 5$, of course the temperature change for the observer depends on both how the temperature changes in space *and* the rate at which the observer is traversing space.

So $\frac{dT}{dt}$ = speed * directional derivative.

(4 points for directional derivative, 4 points for chain rule, 2 points for correct ratio)